

Computing Adem Cohomology Operations

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Abstract

We deal with the problem of obtaining explicit simplicial formulae defining the classical Adem cohomology operations at the cochain level. Having these formulae at hand, we design an algorithm for computing these operations for any finite simplicial set.

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1 Introduction

Cohomology operations are algebraic operations on the cohomology groups of spaces. This machinery is useful when the (co)homology and the cup product on cohomology fail to distinguish two spaces. In the literature, there exists classical methods for computing the (co)homology of spaces (see [Mun84]). Our aim is the computation of cohomology operations on any finite simplicial set. Steenrod squares [Ste47], Steenrod reduced powers [Ste52] and Adem secondary cohomology operations [Ade52] constitute important classes of cohomology operations.

In [GR98, GR99a, GR02b, GR03], the classical definition of Steenrod cohomology operations are reinterpreted in terms of permutations and contractions (special type of homotopy equivalences) of explicit chains for obtaining

explicit formulae for these operations at the cochain level. The underlying combinatorial structures of these formulae are also studied in detail. In [GRb], this point of view is extended to a general theory for obtaining explicit formulae for any cohomology operation that can be expressed at the cochain level. [GRa] is devoted to give general techniques for simplifying the underlying simplicial structures of the formulae obtained using the process explained in [GRb].

In this paper, we deal with the problem of the computation of Adem cohomology operations. For this task, we apply the machinery developed in [GRb] to this special case in order to obtain explicit formulae at the cochain level of these operations. Some properties are also given in order to obtain simplicial “economical” formulae in the sense that they are expressed only in terms of face operators. Finally, since an explicit contraction of the cochain complex to the cohomology can be constructed [GR03], an algorithm for computing Adem cohomology operations at the cohomology level for any finite simplicial set is designed in detail.

The organization of the paper follows. In Section 2, we introduce the theoretical background from Algebraic Topology on the concepts we use here. In Section 3, we give a procedure for obtaining explicit formulae for the morphisms \mathcal{E}_{3i+3} that are involved in the definition of Adem cohomology operations at the cochain level. In Section 4, we design an algorithm for computing Adem cohomology operations using the explicit formulae obtained before.

2 Background

We introduce the basic notation and terminology we use throughout this paper. References for the material in this section are [May67, McL75].

A *simplicial set* K is a graded set indexed by the non-negative integers together with *face* and *degeneracy operators* $\partial_i : K_q \rightarrow K_{q-1}$ and $s_i : K_q \rightarrow K_{q+1}$, $0 \leq i \leq q$, satisfying the following identities:

$$\partial_i \partial_j = \partial_{j-1} \partial_i \text{ if } i < j,$$

$$s_i s_j = s_{j+1} s_i \text{ if } i \leq j,$$

$$\partial_i s_j = s_{j-1} \partial_i \text{ if } i < j, \quad \partial_j s_j = 1_K = \partial_{j+1} s_j, \quad \partial_i s_j = s_j \partial_{i-1} \text{ if } i > j+1.$$

The elements of K_q are called q -simplices. A simplex x is *degenerate* if $x = s_i y$ for some simplex y and degeneracy operator s_i ; otherwise, x is *non-degenerate*.

The *cartesian product* $K \times L$ is a simplicial set whose simplices and face and degeneracy operators are given by $(K \times L)_q = K_q \times L_q$, $\partial_i(x, y) = (\partial_i x, \partial_i y)$ and $s_i(x, y) = (s_i x, s_i y)$. The *semi-direct product* $\mathcal{G} \times_\chi \mathcal{G}'$ is the group $\mathcal{G} \times \mathcal{G}'$ with the group-operation $(g_1, g'_1) \cdot (g_2, g'_2) = (g_1 \cdot \chi(g'_1, g_2), g'_1 \cdot g'_2)$, where $\chi : \mathcal{G}' \otimes \mathcal{G} \rightarrow \mathcal{G}$ satisfies that $\chi(\bar{0}, g_1) = g_1$, $\chi(g'_1, \chi(g'_2, g_1)) = \chi(g'_1 \cdot g'_2, g_1)$ and $\chi(g'_1, g_1 \cdot g_2) = \chi(g'_1, g_1) \cdot \chi(g'_1, g_2)$.

Let R be a commutative ring with identity $1 \neq 0$. A *chain complex* is a graded R -module $C = \bigoplus_{n \in \mathbf{Z}} C_n$ together with an R -module endomorphism of degree -1 , $d = \sum_{n \in \mathbf{Z}} d_n : C_n \rightarrow C_{n-1}$, such that $dd = d^2$ is zero. The map d is called the *differential* of C . $\text{Ker } d_n$ is the module of n -cycles in C ; $\text{Im } d_{n+1}$ is the module of n -boundaries in C ; the quotient $H_n(C) = \text{Ker } d_n / \text{Im } d_{n+1}$ is the n^{th} *homology module* of C . The homology class of a cycle $a \in \text{Ker } d_n$ is denoted by $[a]$. The n^{th} homology of C with coefficients in a ring G is defined by $H_n(C; G) = H_n(C \otimes G)$. Whenever two graded objects x and y of degree p and q are interchanged we apply *Koszul convention* and introduce the sign $(-1)^{pq}$. The *tensor product* of chain complexes C and D is the chain complex $C \otimes D$ with differential $d_{C \otimes D} = d_C \otimes 1_D + 1_C \otimes d_D$. Thus if $x \in C_q$ and $y \in D_r$, an application of Koszul convention gives $d_{C \otimes D}(x \otimes y) = d_C(x) \otimes y + (-1)^q x \otimes d_D(y)$. A module homomorphism $f : C \rightarrow D$ of degree zero such that $df = fd$ is a *chain map*. If $f : C \rightarrow D$ and $g : C' \rightarrow D'$ are chain maps, so is $f \otimes g : C \otimes C' \rightarrow D \otimes D'$.

Let C be a chain complex and G an R -module. Form the abelian group $C^n = \text{Hom}_R(C_n, G)$, for all n ; its elements are the module homomorphisms $c : C_n \rightarrow G$, called n -cochains of C^* . The differential $d : C \rightarrow C$ induces a map of degree $+1$, $\delta : C^* \rightarrow C^*$, defined by $\delta^n c = (-1)^{n+1} c d_{n+1} : C_{n+1} \rightarrow G$, for all $c \in C^n$ and for all n . The *cohomology* of C is the family of abelian groups $H^n(C, G) = \text{Ker } \delta^n / \text{Im } \delta^{n-1}$. An element of $\text{Im } \delta^{n-1}$ is called an n -coboundary and an element of $\text{Ker } \delta^n$ an n -cocycle.

The chain complex of a simplicial set K with coefficients in R , denoted by $C(K)$ is constructed as follows. Let $C_n(K)$ denote the free R -module on the set K_n . The face operators ∂_i yield module maps $C_n(K) \rightarrow C_{n-1}(K)$, which we also call ∂_i ; their alternating sum $d = \sum_{i \geq 0} (-1)^i \partial_i$ is the differential of $C(K)$. The normalized chain complex $C^N(K)$ is the chain complex defined as the quotient $C_n^N(K) = C_n(K) / s(C_{n-1}(K))$, where $s(C_{n-1}(K))$ denotes the free R -module on the set of all degenerate n -simplices of K .

Since we will always work with normalized chain complexes, we simplify notation and write $C(K)$ instead of $C^N(K)$. The (co)homology of K is, by definition, the (co)homology of $C(K)$. The cohomology of K is an algebra with the *cup product* $\smile: H^i(K; G) \otimes H^j(K; G) \rightarrow H^{i+j}(K; G)$ defined by $[c] \smile [c'] = [c \smile c']$, where G is a group, $c \in \text{Ker } \delta^i$, $c' \in \text{Ker } \delta^j$ and $c \smile c'(x) = \mu(c(\partial_{i+1} \cdots \partial_{i+j} x) \otimes c'(\partial_0 \cdots \partial_{i-1} x))$, with $x \in C_{i+j}(K)$ and μ being the operation on G . The following chain maps are needed to be defined:

- The *diagonal map* $\Delta: C(K) \rightarrow C(K^{\times n})$ is defined by $\Delta x = (x, \overset{n \text{ times}}{, x})$.
- The *cyclic permutations*
 $t: C(K^n) \rightarrow C(K^{\times n})$ such that $t(x_1, x_2, \dots, x_n) = (x_2, \dots, x_n, x_1)$;
and $T: C(K)^{\otimes n} \rightarrow C(K)^{\otimes n}$ defined by
 $T(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = (-1)^{|x_1|(|x_2| + \cdots + |x_n|)} x_2 \otimes \cdots \otimes x_n \otimes x_1$.

A *differential graded module* (DG-module, for short) M is a chain complex such that $M_n = 0$ for all $n < 0$. A *DGA-module* (M, ξ, η) (we will write it simply M when no confusion can arise) is a DG-module M endowed with two morphisms called the *augmentation* $\xi: M_0 \rightarrow R$ and the *coaugmentation*, $\eta: R \rightarrow M_0$. It is required that $\xi \eta = 1_R$ and $\xi d = 0$.

A *DGA-algebra* (A, μ) (resp. *DGA-coalgebra* (B, ∇)) is a DGA-module endowed with a morphism $\mu: A \otimes A \rightarrow A$, called *product* on A , such that $\mu(\mu \otimes 1_A) = \mu(1_A \otimes \mu)$ and $\mu(\eta_A \otimes 1_A) = 1_A = \mu(1_A \otimes \eta_A)$. Resp. $\nabla: B \rightarrow B \otimes B$, called *coproduct* on B , where $(\nabla \otimes 1_B)\nabla = (1_B \otimes \nabla)\nabla$ and $(\eta_B \otimes 1_B)\nabla = 1_B = (1_B \otimes \eta_B)\nabla$.

The free R -algebra generated by a group \mathcal{G} is a DGA-algebra denoted by $(R[\mathcal{G}], \xi_{\mathcal{G}}, \eta_{\mathcal{G}}, \mu_{\mathcal{G}})$ such that it is zero in each degree except for degree zero, where $R_0[\mathcal{G}] = \{\sum_{a \in A} \lambda_a a : \lambda_a \in R \text{ and } A \text{ is a finite subset of } \mathcal{G}\}$. The product $\mu_{\mathcal{G}}$, the augmentation $\xi_{\mathcal{G}}$ and the coaugmentation $\eta_{\mathcal{G}}$ are given by $\mu_{\mathcal{G}}((\sum \lambda_a a) \otimes (\sum \lambda_{a'} a')) = \sum \lambda_a \lambda_{a'} (a + a')$, $\xi_{\mathcal{G}}(\sum \lambda_a a) = \sum \lambda_a$ and $\eta_{\mathcal{G}}(\lambda) = \lambda \bar{0}$, where $a, a' \in \mathcal{G}$ and $\lambda_a, \lambda_{a'}, \lambda \in R$.

The *reduced bar construction* $\bar{B}(\mathcal{G})$ of the DGA-algebra $R[\mathcal{G}]$ is defined (as a graded module) in degree $n > 0$ by $\bar{B}_n(\mathcal{G}) = R[\mathcal{G}]^{\otimes n} / s(R[\mathcal{G}]^{\otimes n})$ where $s(R[\mathcal{G}]^{\otimes n})$ is the R -module generated by all the elements of $R[\mathcal{G}]^{\otimes n}$ of the form $a_1 \otimes \cdots \otimes \bar{0} \otimes \cdots \otimes a_n$; and $\bar{B}_0(\mathcal{G}) = R$. The element of $\bar{B}_0(\mathcal{G})$ corresponding to the identity in R is denoted by $[\]$ and an element $a_1 \otimes \cdots \otimes a_n$ of $\bar{B}_n(\mathcal{G})$ is denoted by $[a_1 | \cdots | a_n]$. The differential of $\bar{B}(\mathcal{G})$ is given by

$$d([a_1 | \cdots | a_n]) = \xi_{\mathcal{G}}(a_1)[a_2 | \cdots | a_n] + (-1)^n [a_1 | \cdots | a_{n-1}] \xi_{\mathcal{G}}(a_n) \\ + \sum_{1 \leq i \leq n-1} (-1)^i [a_1 | \cdots | a_{i-1} | \mu_{\mathcal{G}}(a_i \otimes a_{i+1}) | a_{i+2} | \cdots | a_n].$$

Observe that $d[a_1] = 0$ for all $[a_1] \in \bar{B}_1(\mathcal{G})$. The augmentation and the coaugmentation on $\bar{B}(\mathcal{G})$ coincide with the identity on R . Moreover, $\bar{B}(\mathcal{G})$ is a DGA-coalgebra with the coproduct:

$$\nabla([a_1 | \cdots | a_n]) = \sum_{0 \leq i \leq n} [a_1 | \cdots | a_i] \otimes [a_{i+1} | \cdots | a_n].$$

Let (B, ∇) be a DGA-coalgebra and (A, μ) a DGA-algebra. A *twisting cochain* κ , is a graded module morphism of degree -1 , $\kappa : B \rightarrow A$, satisfying that $d_A \kappa + \kappa d_B + \mu(\kappa \otimes \kappa) \nabla = 0$, $\xi_A \kappa = 0$ and $\kappa \eta_B = 0$. Let M be an A -DG-module (where $\nu : M \otimes A \rightarrow M$ is the (right) A -module structure on A). Define the morphism $d_\kappa : M \otimes B \rightarrow M \otimes B$ where $d_\kappa(m \otimes b) = (d_{M \otimes B} + \kappa \cap)(m \otimes b)$ and $\kappa \cap = (\nu \otimes 1_B)(1_M \otimes \kappa \otimes 1_B)(1_M \otimes \nabla)$. The graded module $M \otimes B$ endowed with d_κ is a DG-module denoted by $M \otimes_\kappa B$ and called *twisted tensor product* by the twisting cochain κ . An example of twisted tensor product is $R[\mathcal{G}] \otimes_\theta \bar{B}(\mathcal{G})$, where the twisting cochain θ called the *universal twisting cochain* is given by $\theta([a_1]) = a_1 - \bar{0}$ and $\theta([a_1 | \cdots | a_n]) = 0$ for $n > 1$.

We deal with an special type of homotopy equivalences. A *contraction* r of a DG-module N to a DG-module M , consists in three morphisms (f, g, ϕ) where $f : N \rightarrow M$ (projection) and $g : M \rightarrow N$ (inclusion) are DG-module morphisms of degree zero, and $\phi : N \rightarrow N$ (homotopy operator) is a morphism of degree -1 satisfying that $fg = 1_M$, $\phi d + d\phi = gf - 1_N$. Moreover, it is required that $\phi g = 0$, $f\phi = 0$, $\phi\phi = 0$. A contraction will be denoted by $r = (f, g, \phi) : N \Rightarrow M$ or briefly $N \xrightarrow{r} M$. Note that the importance of having this structure of M to N is that N is “smaller” than M although both have the same homology. Let $r = (f, g, \phi) : N \Rightarrow M$ and $r' = (f', g', \phi') : N' \Rightarrow M'$ be two contractions, then the following contractions can be constructed:

- The tensor product contraction: $r \otimes r' = (f \otimes f', g \otimes g', \phi \otimes g'f' + 1_N \otimes \phi') : N \otimes N' \Rightarrow M \otimes M'$.
- If $M = N'$, the composition contraction: $r'r = (f'f, gg', \phi + g\phi'f) : N \Rightarrow M'$.

Let p and q be non-negative integers, a (p, q) -shuffle (α, β) is a partition of the set $\{0, 1, \dots, p + q - 1\}$ in two disjoint subsets, $\alpha_1 < \dots < \alpha_p$ and $\beta_1 < \dots < \beta_q$, of p and q integers, respectively. The signature of the shuffle (α, β) is defined by $sig(\alpha, \beta) = \sum_{1 \leq i \leq p} \alpha_i - (i - 1)$.

An Eilenberg–Zilber contraction [EZ59] of $C(K \times L)$ to $C(K) \otimes C(L)$, where K and L are simplicial sets, is a triple $r_{EZ} = (Aw, Em, Sh)$ where:

- Alexander–Whitney operator $Aw : C(K \times L) \rightarrow C(K) \otimes C(L)$ is defined by: $Aw(x_m \times y_m) = \sum_{0 \leq i \leq m} \partial_{i+1} \cdots \partial_m x_m \otimes \partial_0 \cdots \partial_{i-1} y_m$.
- Eilenberg–Mac Lane operator $Em : C(K) \otimes C(L) \rightarrow C(K \times L)$ by:

$$Em(x_p \otimes y_q) = \sum_{(\alpha, \beta) \in \{(p, q)\text{-shuffles}\}} (-1)^{sig(\alpha, \beta)} (s_\beta x_p, s_\alpha y_q)$$
where $s_\beta = s_{\beta_q} \cdots s_{\beta_1}$ and $s_\alpha = s_{\alpha_p} \cdots s_{\alpha_1}$.
- Shih operator $Sh : C(K \times L) \rightarrow C(K \times L)$ by:

$$Sh((x_m, y_m)) = \sum_{T(m)} (-1)^{sg} (s_{\bar{\beta} + \bar{m}} \partial_{m-q+1} \cdots \partial_m x_m, s_{\alpha + \bar{m}} \partial_{\bar{m}} \cdots \partial_{m-q-1} y_m)$$
where $sg = \bar{m} + sig(\alpha, \beta) + 1$, $\bar{m} = m - p - q$, $\bar{\beta} + \bar{m} = \{\beta_q + \bar{m}, \dots, \beta_1 + \bar{m}, \bar{m} - 1\}$, $\alpha + \bar{m} = \{\alpha_{p+1} + \bar{m}, \dots, \alpha_1 + \bar{m}\}$ and $T(m) = \{0 \leq p \leq m - q - 1 \leq m - 1, (\alpha, \beta) \in \{(p + 1, q)\text{-shuffles}\}\}$.

It is evident that Alexander–Whitney operator has a polynomial nature. However, Eilenberg–MacLane and Shih operator have an essential “exponential” character because shuffles of degeneracy operators are involved in their respective formulation. A recursive formula for the Shih operator is given in [EM54]. An explicit formula for this operator was stated by J. Rubio [Rub91] and proved by F. Morace [Rea00].

It is possible to construct a contraction $r_{EZ(p)} = (Aw_p, Em_p, Sh_p)$ of $C(K^{\times p})$ to $C(K)^{\otimes p}$ ($p \geq 0$), appropriately composing Eilenberg–Zilber contractions.

Note that the cup product defined on page 4 can be written as $c \smile c' = \mu(c \otimes c') Aw \Delta$ where μ is the operation defined on G . Generalizations of the cup product are the cochain mappings called *cup- i product* [Ste47], $\smile_i : C^n(K; \mathbf{Z}) \otimes C^m(K; \mathbf{Z}) \rightarrow C^{n+m-i}(K; \mathbf{Z})$ given by $c \smile_i c' = \mu(c \otimes c') D_i \Delta$ where $D_i = Aw(t Sh)^i$. The following relation holds (up to sign):

$$\delta(c \smile_i c') = \pm c \smile_{i-1} c' \pm c' \smile_{i-1} c + \delta c \smile_i c' + c \smile_i \delta c'. \quad (1)$$

Taking $[c] \in H^j(K; \mathbf{Z})$ and defining $Sq^i[c] = [c \smile_{j-i} c]$, then $Sq^i(c) \in H^{j-i}(K; \mathbf{Z}_2)$. These cohomology operations are called *Steenrod squares* [Ste47].

Explicit formulae for D_i and therefore for computing cup- i products, Steenrod squares and Steenrod reduced powers [Ste52] (a generalization of Steenrod squares), can be found in [GR98, GR99a, GR99b, GR02b].

In [Ade52], J. Adem constructed secondary cohomology operations using relations on iterated Steenrod squares. He proved the relation $Sq^2 Sq^2 + Sq^3 Sq^1 = 0$ by means of the existence of cochain mappings $\mathcal{E}_j : C^p(K^{\times 4}; \mathbf{Z}) \rightarrow C^{p-j}(K; \mathbf{Z}_2)$ such that mod 2

$$(c \smile_i c) \smile_{i+2} (c \smile_i c) + (c \smile_{i+1} c) \smile_i (c \smile_{i+1} c) = \delta \mathcal{E}_{3i+3}(c^4) \quad (2)$$

where \smile_k is the cup- k product, c is a q -cocycle and $i = q - 2$. If c is a q -cocycle such that $Sq^2(c)$ is a coboundary (that is, there exists a cochain b such that $c \smile_i c = \delta b$), then

$$w = b \smile_{i+1} b + b \smile_{i+2} \delta b + \mathcal{E}_{3i+3}(c^4) + \eta(c) \smile_{i-1} \eta(c) + \eta(c) \smile_i \delta \eta(c)$$

is a mod 2 cocycle, where $\eta(c) = \frac{1}{2}(c \smile_{i+2} c) + c$. Adem secondary cohomology operations are defined as $\Psi_q[c] = [w] + Sq^2 H^{q+1}(K; \mathbf{Z}) \in H^{q+3}(K; \mathbf{Z}_2)$.

In Section 4 we give a procedure for computing the Adem secondary cohomology operation $\Psi_q(\alpha)$ for any integer q and any cohomology class α . For doing this, the obtention of explicit formulae for \mathcal{E}_{3i+3} is essential. We will obtain these formulae in the following section.

3 Adem Cocyclic Operations

In order to obtain “economical” formulae for \mathcal{E}_{3i+3} in the sense that they are written only in terms of face operators, we first write \mathcal{E}_{3i+3} in terms of the component morphisms of Eilenberg–Zilber contractions. After this, we make a simplification of the given formulae based on the fact that any composition of face and degeneracy operators can be put in a unique form.

For the first aim, we will use the following result whose demonstration is given in [GRb].

Lemma 1 *Let \mathcal{G} be a group. Let M and N be two $R[\mathcal{G}]$ -DG-modules, where $\nu : M \otimes R[\mathcal{G}] \rightarrow M$ and $\nu' : N \otimes R[\mathcal{G}] \rightarrow N$ are the (right) $R[\mathcal{G}]$ -module structures on $R[\mathcal{G}]$. Let $r = (f, g, \phi) : M \rightrightarrows N$ be a contraction such that $(g \otimes 1_{R[\mathcal{G}]})\nu' = \nu g$.*

Then $(r \otimes 1_{\bar{B}[\mathcal{G}]})_{\theta \cap} = ((f \otimes 1_{\bar{B}[\mathcal{G}]})_{\theta \cap}, g \otimes 1_{\bar{B}[\mathcal{G}]}, (\phi \otimes 1_{\bar{B}[\mathcal{G}]})_{\theta \cap})$ of $M \otimes_{\theta} \bar{B}(\mathcal{G})$ to $N \otimes_{\theta} \bar{B}(\mathcal{G})$ is a new contraction where $(f \otimes 1_{\bar{B}[\mathcal{G}]})_{\theta \cap} = \sum_{i \geq 0} (f \otimes 1_{\bar{B}[\mathcal{G}]})_{\theta \cap} (\theta \cap (\phi \otimes 1_{\bar{B}[\mathcal{G}]})^i)$ and $(\phi \otimes 1_{\bar{B}[\mathcal{G}]})_{\theta \cap} = - \sum_{i \geq 0} ((\phi \otimes 1_{\bar{B}[\mathcal{G}]})_{\theta \cap})^i (\phi \otimes 1_{\bar{B}[\mathcal{G}]})$.

Now, let \mathcal{G} be the semi-direct product $\mathbf{Z}_2^{\times 2} \times_{\chi} \mathbf{Z}_2$ where $\chi((a, b), \bar{1}) = (b, a)$. Let $M = C(K^{\times 4})$ and $N = C(K)^{\otimes 4}$ be two $\mathbf{Z}_2[\mathcal{G}]$ -modules:

$$\begin{aligned} \nu(x, a_1) &= z(x) = (x_1, x_3, x_2, x_4), & \nu'(y, a_1) &= z'(y) = y_1 \otimes y_3 \otimes y_2 \otimes y_4, \\ \nu(x, a_2) &= t^{\times 2}(x) = (x_2, x_1, x_4, x_3), & \nu'(y, a_2) &= T^{\otimes 2}(y) = y_2 \otimes y_1 \otimes y_4 \otimes y_3, \\ \nu(x, a_3) &= t(x) = (x_3, x_4, x_1, x_2), & \nu'(y, a_3) &= T(y) = y_3 \otimes y_4 \otimes y_1 \otimes y_2; \end{aligned}$$

where $a_1 = ((\bar{0}, \bar{0}), \bar{1})$, $a_2 = ((\bar{1}, \bar{0}), \bar{0})$, $a_3 = ((\bar{0}, \bar{1}), \bar{0})$, $x = (x_1, x_2, x_3, x_4) \in C(K^{\times 4})$ and $y = y_1 \otimes y_2 \otimes y_3 \otimes y_4 \in C(K)^{\otimes 4}$. Let r be the Eilenberg–Zilber contraction $r_{EZ(4)} = (r_{EZ}^{\otimes 2})r_{EZ} = (Aw_4, Em_4, Sh_4) : C(K^{\times 4}) \Rightarrow C(K)^{\otimes 4}$ which commutes with the structures given above. Observe that $Aw_4 = (Aw^{\otimes 2})Aw$, $Em_4 = Em(Em^{\otimes 2})$ and $Sh_4 = Sh + Em(Sh \otimes EmAw + 1 \otimes Sh)Aw$. Lemma 1 produces the new contraction:

$$(r_{EZ(4)} \otimes 1_{\bar{B}(\mathbf{Z}_2^{\times 2} \times_{\chi} \mathbf{Z}_2)})_{\theta \cap} : C(K^{\times 4}) \otimes_{\theta} \bar{B}(\mathbf{Z}_2^{\times 2} \times_{\chi} \mathbf{Z}_2) \Rightarrow C(K)^{\otimes 4} \otimes_{\theta} \bar{B}(\mathbf{Z}_2^{\times 2} \times_{\chi} \mathbf{Z}_2).$$

As we will see now, Adem relation (2) can be obtained from the fact that the projection $(Aw_4 \otimes 1_{\bar{B}(\mathcal{G})})_{\theta \cap}$ is a DG-module morphism.

Theorem 2 *Adem relation (2) is obtained from the identity:*

$$\begin{aligned} &\mu c^{\otimes 4} (1_N \otimes \xi_{\bar{B}(\mathcal{G})}) (Aw_4 \otimes 1_{\bar{B}(\mathcal{G})})_{\theta \cap} (d_{M \otimes \bar{B}(\mathcal{G})} + \theta \cap) (\Delta(x) \otimes e_{3i+3}) \\ &= \mu c^{\otimes 4} (1_N \otimes \xi_{\bar{B}(\mathcal{G})}) (d_{N \otimes \bar{B}(\mathcal{G})} + \theta \cap) (Aw_4 \otimes 1_{\bar{B}(\mathcal{G})})_{\theta \cap} (\Delta(x) \otimes e_{3i+3}) \end{aligned}$$

where μ is the product on \mathbf{Z}_2 , $c \in \text{Ker } \delta^q$, $x \in C_{q+3}(K)$ and $e_{3i+3} \in \bar{B}(\mathcal{G})$ is:

$$\begin{aligned} e_{3i+3} &= e_{(3,0)} \quad \text{for } i = 0, \\ e_{3i+3} &= e_{(3i+3,i)} + e_{(3i+3,i-1)} \quad \text{for } i = 2, 4, \\ e_{3i+3} &= e_{(3i+3,i)} \quad \text{for } i = 1, 3, 5 \text{ and for } i \geq 6, i \text{ being odd and } \frac{i-7}{2} \text{ being even,} \\ e_{3i+3} &= e_{(3i+3,i)} + e_{(3i+3,i-2)} \quad \text{for } i \geq 6, i, \frac{i-7}{2} \text{ being odd,} \\ e_{3i+3} &= e_{(3i+3,i)} + e_{(3i+3,i-1)} + e_{(3i+3,i-2)} \quad \text{for } i \geq 6, i \text{ being even,} \end{aligned}$$

such that

$$e_{(3i+3,\ell)} = \sum_{S(3i+2,\ell)} [(c_{\pi_1}, \bar{0}) | \dots | (c_{\pi_j}, \bar{0}) | ((\bar{0}, \bar{0}), \bar{1}) | (\bar{c}_{\pi_{j+1}}, \bar{0}) | \dots | (\bar{c}_{\pi_{3i+2}}, \bar{0})]$$

where $S(3i+2, \ell) = \{\pi \in \{(3i+2-\ell, \ell)\text{-sh.}\}\} \cup \{j : 0 \leq j \leq 3i+2\}$,
 $[c_1, \dots, c_{3i+2}] = [(\bar{1}, \bar{0}), 3i+2-\ell, (\bar{1}, \bar{0}), (\bar{0}, \bar{1}), \dots, (\bar{0}, \bar{1})]$ and if $c = (a, b)$ then
 $\bar{c} = (b, a)$.

The proof of this theorem is given in an appendix at the end of this paper.

Corollary 3 *The explicit formulae for \mathcal{E}_{3i+3} are:*

$$\mathcal{E}_3 = \tilde{\mathcal{E}}_{(3,0)} + c \smile_0 c \smile_1 c \smile_2 c \text{ for } i = 0.$$

$$\mathcal{E}_6 = \tilde{\mathcal{E}}_{(6,1)} + c \smile_1 c \smile_2 c \smile_3 c \text{ for } i = 1.$$

$$\mathcal{E}_9 = \tilde{\mathcal{E}}_{(9,2)} + \tilde{\mathcal{E}}_{(9,1)} + c \smile_2 c \smile_3 c \smile_4 c + c \smile_0 c \smile_7 c \smile_2 \text{ for } i = 2.$$

$$\mathcal{E}_{12} = \tilde{\mathcal{E}}_{(12,3)} + c \smile_3 c \smile_4 c \smile_5 c + c \smile_2 c \smile_7 c \smile_3 c + c \smile_3 c \smile_8 c \smile_1 c \text{ for } i = 3.$$

$$\mathcal{E}_{15} = \tilde{\mathcal{E}}_{(15,4)} + \tilde{\mathcal{E}}_{(15,3)} + c \smile_4 c \smile_5 c \smile_6 c \text{ for } i = 4.$$

$$\mathcal{E}_{18} = \tilde{\mathcal{E}}_{(18,5)} + c \smile_5 c \smile_6 c \smile_7 c + c \smile_4 c \smile_9 c \smile_5 c + c \smile_4 c \smile_{11} c \smile_3 c \text{ for } i = 5.$$

$$\mathcal{E}_{3i+3} = \tilde{\mathcal{E}}_{(3i+3,i)} + \tilde{\mathcal{E}}_{(3i+3,i-1)} + \tilde{\mathcal{E}}_{(3i+3,i-2)} + c \smile_i c \smile_{i+1} c \smile_{i+2} \text{ for } i \geq 6, i \text{ being even, } \frac{i-6}{2} \text{ being odd.}$$

$$\mathcal{E}_{3i+3} = \tilde{\mathcal{E}}_{(3i+3,i)} + \tilde{\mathcal{E}}_{(3i+3,i-1)} + \tilde{\mathcal{E}}_{(3i+3,i-2)} + c \smile_i c \smile_{i+1} c \smile_{i+2} + c \smile_i c \smile_{i+5} c \smile_{i-2} \text{ for } i \geq 6, i, \frac{i-6}{2} \text{ being even.}$$

$$\mathcal{E}_{3i+3} = \tilde{\mathcal{E}}_{(3i+3,i)} + \tilde{\mathcal{E}}_{(3i+3,i-2)} + c \smile_i c \smile_{i+1} c \smile_{i+2} + c \smile_i c \smile_{i+4} c \smile_{i-1} + c \smile_i c \smile_{i+5} c \smile_{i-2} + c \smile_{i-1} c \smile_{i+6} c \smile_{i-2} \text{ for } i \geq 7, i, \frac{i-7}{2} \text{ being odd.}$$

$$\mathcal{E}_{3i+3} = \tilde{\mathcal{E}}_{(3i+3,i)} + c \smile_i c \smile_{i+1} c \smile_{i+2} + c \smile_i c \smile_{i+4} c \smile_{i-1} + c \smile_i c \smile_{i+5} c \smile_{i-2} + c \smile_{i-1} c \smile_{i+6} c \smile_{i-2} \text{ for } i \geq 7, i \text{ being odd, } \frac{i-7}{2} \text{ being even.}$$

where $\tilde{\mathcal{E}}_{(3i+3,\ell)}$ denotes the composition:

$$\begin{aligned} & \sum \mu c^{\otimes 4} (Aw_4(t Sh_4)^{m_0} (t^{\times 2} Sh_4)^{n_1} (t Sh_4)^{m_1} \dots z Sh_4 \dots (t^{\times 2} Sh_4)^{n_j} (t Sh_4)^{m_j} \\ & + Aw_4(t^{\times 2} Sh_4)^{m_0} (t Sh_4)^{n_1} (t^{\times 2} Sh_4)^{m_1} \dots z Sh_4 \dots (t Sh_4)^{n_j} (t^{\times 2} Sh_4)^{m_j} \Delta(x) \end{aligned} \quad (3)$$

such that the sum is taken over the set $\{(n_1, \dots, n_j, m_0, \dots, m_j) : n_1 + \dots + n_j = 3i + 2 - \ell, m_0 + \dots + m_j = \ell, n_k \geq 0, m_k \geq 0\}$.

Observe that an algorithm designed for computing Adem operations from these formulae would be too slow for practical implementation because the morphisms Sh_4 and Em_4 present an exponential number of summands in their formulae. For this reason, the idea of simplification arises in a natural way. In the following subsection we give some useful properties for the normalization process. Some work have been done in this way by the authors in [GRa].

3.1 Adem Cocyclic Operations and Combinatorics

A refinement of \mathcal{E}_{3i+3} , only in terms of face operators of K , is feasible using a normalization process. The following properties will be useful in this process.

Proposition 4 *Let K be a simplicial set then:*

1. *Any composition of face and degeneracy operators of K can be put in the unique form: $s_{j_t} \dots s_{j_1} \partial_{i_1} \dots \partial_{i_s}$, where $j_t > \dots > j_1 \geq 0$ and $i_s > \dots > i_1 \geq 0$.*
2. *Those summands of the simplified formula for \mathcal{E}_{3i+3} with a factor having a degeneracy operator in its expression are null.*
3. *All the summands of $(1 \otimes Sh)Awz Sh : C(K^{\times 4}) \rightarrow C(K^{\times 2})^{\otimes 2}$ are null.*
4. *$Awz Em = (Em \otimes Em)z'(Aw \otimes Aw)$ where $Awz Em : C(K^{\times 2})^{\otimes 2} \rightarrow C(K^{\times 2})^{\otimes 2}$.*
5. *It is satisfied that $Sh t Sh = Sh t \tilde{Sh} : C(K^{\times 2}) \rightarrow C(K^{\times 2})$ where $\tilde{Sh} : C(K^{\times 2}) \rightarrow C(K^{\times 2})$ consists in all the summands of Sh with $\beta_q < \alpha_1$.*

Let us prove the third assertion. Let $(x_1, x_2, y_1, y_2) \in C_m(K^{\times 4})$ then the second factor of each summand of $(1 \otimes Sh)Awz Sh(x_1, x_2, y_1, y_2)$ has the form

$$(s_{\bar{b}+\bar{n}}\partial_{n-q'+1}\cdots\partial_n\partial_0\cdots\partial_{i-1}s_{\bar{\beta}+\bar{m}}\partial_{m-q+1}\cdots\partial_mx_2, \\ s_{a+\bar{n}}\partial_{\bar{n}}\cdots\partial_{n-q'-1}\partial_0\cdots\partial_{i-1}s_{\alpha+\bar{m}}\partial_{\bar{m}}\cdots\partial_{m-q-1}y_2)$$

where $n = m - i + 1$, $0 \leq i \leq m + 1$, $0 \leq p \leq m - q - 1 \leq m - 1$, $0 \leq p' \leq n - q' - 1 \leq n - 1$, $(\alpha, \beta) \in \{(p + 1, q)\text{-sh.}\}$ and $(a, b) \in \{(p' + 1, q')\text{-sh.}\}$.

If $i = m + 1$ then $n = 0$, and Sh is 0 in degree 0. When $0 \leq i \leq m$ then $\alpha + \bar{m} \cap \{m - q' + 1, m - q' + 2, \dots, m\} = \emptyset$ because in other case the corresponding cartesian product is degenerate. Then $\{m - q' + 1, m - q' + 2, \dots, m\} \subset \bar{\beta} + \bar{m}$. For the same reason than above, $\bar{\beta} + \bar{m} \cap \{m - p' - q', m - p' - q' + 1, \dots, m - q'\} = \emptyset$, and it follows that $\{m - p' - q', m - p' - q' + 1, \dots, m - q'\} \subset \alpha + \bar{m}$ and $\bar{m} - 1 < m - p' - q'$. But this case is also degenerate.

We leave it to the reader to verify the other assertions because they follow the same technique.

For example, the normalized formula for \mathcal{E}_3 is:

$$\begin{aligned} \mathcal{E}_3(c^4)(x) = & \mu c^{\otimes 4}(\partial_1\partial_4\partial_5 \otimes \partial_3\partial_4\partial_5 \otimes \partial_0\partial_1\partial_2 \otimes \partial_0\partial_1\partial_4 \\ & + \partial_1\partial_2\partial_3 \otimes \partial_0\partial_1\partial_2 \otimes \partial_3\partial_4\partial_5 \otimes \partial_3\partial_4\partial_5 + \partial_2\partial_3\partial_4 \otimes \partial_0\partial_1\partial_2 \otimes \partial_0\partial_4\partial_5 \otimes \partial_0\partial_4\partial_5 \\ & + \partial_3\partial_4\partial_5 \otimes \partial_0\partial_1\partial_3 \otimes \partial_0\partial_1\partial_5 \otimes \partial_0\partial_1\partial_5 + \partial_3\partial_4\partial_5 \otimes \partial_0\partial_1\partial_4 \otimes \partial_0\partial_1\partial_2 \otimes \\ & \partial_0\partial_1\partial_2)\Delta(x) \end{aligned}$$

where c is a 2-cocycle, x is a 5-simplex and μ is the product on \mathbf{Z}_2 .

4 Adem Cohomology Operations

An algorithm for computing the homology (if it is torsion free or the ground ring is a field) of a finite simplicial set K and a contraction (f, g, ϕ) of $C(K)$ to $H(K)$ appear in [GR03]. The complexity of the algorithm is $O(m^3)$, where m is the number of simplices of K . From this contraction, it is easy to derive another one, (f^*, g^*, ϕ^*) , of $C^*(K; \mathbf{Z})$ to $H^*(K; \mathbf{Z})$ (see [GR03]). An interesting property of this contraction is that if c is a coboundary, then $c = \delta\phi^*(c)$.

For attacking the computation of Adem secondary cohomology operations, we will see that the homotopy operator ϕ^* , the explicit formulae for computing cup- i products given in [GR99a] and the explicit formulae for \mathcal{E}_{3i+3} are essential. The steps for computing Ψ_q are the following.

Algorithm 5 Algorithm for computing Adem cohomology operations.

INPUT: A finite simplicial set K and an integer $q = i - 2 \geq 2$.

1. Compute (f^*, g^*, ϕ^*) of $C^*(K; \mathbf{Z})$ to $H^*(K; \mathbf{Z})$.
Let $\{\alpha_1, \dots, \alpha_p\}$ be a set of generators of $H^q(K; \mathbf{Z})$.
2. Compute $Sq^2 : H^q(K; \mathbf{Z}) \rightarrow H^{q+2}(K; \mathbf{Z}_2)$ as follows:
For $j = 1$ to $j = p$, compute $f^*(g^*(\alpha_j) \smile_i g^*(\alpha_j))$.
3. Put the matrix corresponding to $Sq^2 : H^q(K; \mathbf{Z}) \rightarrow H^{q+2}(K; \mathbf{Z}_2)$ in a diagonal form D .
Let $\{\beta_1, \dots, \beta_r\}$ be a set of generators of $\text{Ker } Sq^2$ obtained using D .
4. For $j = 1$ to $j = r$, compute $c_j = g^*(\beta_j)$, $b_j = \phi^*(c_j \smile_i c_j)$ and $\eta(c_j) = \frac{1}{2}(c_j \smile_{i+2} c_j + c_j)$.
5. For $j = 1$ to $j = r$, compute $w_j = b_j \smile_{i+1} b_j + b_j \smile_{i+2} \delta(b_j) + \mathcal{E}_{3i+3}(c_j^4) + \eta(c_j) \smile_{i-1} \eta(c_j) + \eta(c_j) \smile_i \delta\eta(c_j)$.
6. For $j = 1$ to $j = r$, compute $f^*(w_j)$.

OUTPUT: $\Psi_q : H^q(K; \mathbf{Z}) \rightarrow H^{q+3}(K; \mathbf{Z}_2)$

A practical implementation in *Mathematica*[©] and a concrete example about the computation of the first Adem operation Ψ_2 is given in [GR02a].

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Appendix: Proof of Theorem 2

First of all, the chain e_{3i+3} ($i \geq 0$) is obtained using the following composition of contractions:

$$\begin{aligned} \bar{B}(\mathbf{Z}_2 \times_{\chi} \mathbf{Z}_2^{\times 2}) &\xrightarrow{r_{EZ(\chi)}} \bar{B}(\mathbf{Z}_2) \otimes_{\theta} \bar{B}(\mathbf{Z}_2^{\times 2}) \xrightarrow{(1 \otimes r_{\bar{B}})_{\theta \cap}} \bar{B}(\mathbf{Z}_2) \otimes_{\theta} \bar{B}(\mathbf{Z}_2)^{\otimes 2} \\ &\quad \downarrow r_2^{\otimes 3} \\ &E(u) \otimes \Gamma(v) \otimes_{\theta} (E(u_1) \otimes \Gamma(v_1) \otimes E(u_2) \otimes \Gamma(v_2)) \end{aligned}$$

where $r_{EZ(\chi)} = (Aw_{\chi}, Em_{\chi}, Sh_{\chi})$ is a perturbed Eilenberg–Zilber contraction such that the inclusion Em_{χ} is not perturbed [Arm99]; $(1 \otimes r_{\bar{B}})_{\theta \cap}$ satisfies that the inclusion is $1 \otimes g_{\bar{B}}$ (Lemma 1) and r_2 is an isomorphism with explicit formulae given in [EM52].

Denote the composition $Em_{\chi}(1 \otimes g_{\bar{B}})g_2^{\otimes 3}$ of $E(u) \otimes \Gamma(v) \otimes_{\theta} (E(u_1) \otimes \Gamma(v_1) \otimes E(u_2) \otimes \Gamma(v_2))$ to $\bar{B}(\mathbf{Z}_2 \times_{\chi} \mathbf{Z}_2^{\times 2})$ by $g_{E\Gamma}$. Let w_i^j ($i=1,2$) be the generator of $E(u_i) \otimes \Gamma(v_i)$ in degree j . Then,

$$\begin{aligned} e_{(3i+3, \ell)} &= g_{E\Gamma}(u \otimes w_1^{\ell} \otimes w_2^{3i+2-\ell}) \\ &= \sum_{S(3i+2, \ell)} [(c_{\pi_1}, \bar{0}) | \dots | (c_{\pi_j}, \bar{0}) | ((\bar{0}, \bar{0}), \bar{1}) | (\bar{c}_{\pi_{j+1}}, \bar{0}) | \dots | (\bar{c}_{\pi_{3i+2}}, \bar{0})]. \end{aligned}$$

We have to prove that Adem relation (2) is obtained from the identity:

$$\begin{aligned} &\mu c^{\otimes 4}(1_N \otimes \xi_{\bar{B}(\mathcal{G})})(Aw_4 \otimes 1_{\bar{B}(\mathcal{G})})_{\theta \cap} (d_{M \otimes \bar{B}(\mathcal{G})} + \theta \cap)(\Delta(x) \otimes e_{3i+3}) \\ &= \mu c^{\otimes 4}(1_N \otimes \xi_{\bar{B}(\mathcal{G})})(d_{N \otimes \bar{B}(\mathcal{G})} + \theta \cap)(Aw_4 \otimes 1_{\bar{B}(\mathcal{G})})_{\theta \cap} (\Delta(x) \otimes e_{3i+3}) \end{aligned} \quad (4)$$

Since e_{3i+3} is defined as a sum of elements of the form $e_{(3i+3, \ell)}$ for $\ell = i, i-1, i-2$, the left-hand side of (4) has the form:

$$\sum_{\ell} \mu c^{\otimes 4}(1_N \otimes \xi_{\bar{B}(\mathcal{G})})(Aw_4 \otimes 1_{\bar{B}(\mathcal{G})})_{\theta \cap} (d_M \otimes 1_{\bar{B}(\mathcal{G})})(\Delta(x) \otimes e_{(3i+3, \ell)}) \quad (5)$$

and the right-hand side, the form:

$$\sum_{\ell} \mu c^{\otimes 4}(1_N \otimes \xi_{\bar{B}(\mathcal{G})})(Aw_4 \otimes 1_{\bar{B}(\mathcal{G})})_{\theta \cap} (\Delta(x) \otimes b_{(3i+2, \ell)}) \quad (6)$$

where

$$\begin{aligned}
b_{(3i+2,\ell)} &= g_{E\Gamma} d_{\bar{B}(\mathbf{Z}_2 \times_{\chi} \mathbf{Z}_2^{\times 2})}(e_{(3i+3,\ell)}) = g_{E\Gamma}(w_1^\ell \otimes w_2^{3i+2-\ell} + w_1^{3i+2-\ell} \otimes w_2^\ell) \\
&= \sum_{S(3i+2,\ell)} [(c_{\pi_1}, \bar{0}) | \dots | (c_{\pi_{3i+2}}, \bar{0})] + [(\bar{c}_{\pi_1}, \bar{0}) | \dots | (\bar{c}_{\pi_{3i+2}}, \bar{0})]
\end{aligned}$$

On one hand, (5) can be simplified to $\delta(\sum_{\ell} \tilde{\mathcal{E}}_{(3i+3,\ell)})$ where $\tilde{\mathcal{E}}_{(3i+3,\ell)}$ denotes the composition (3) defined on page 10.

On the other hand, (6) can be written mod 2 as

$$\begin{aligned}
&\sum_{\ell} \sum \mu c^{\otimes 4} (Aw_4(t Sh_4)^{m_0} (t^{\times 2} Sh_4)^{n_1} (t Sh_4)^{m_1} \dots (t^{\times 2} Sh_4)^{n_j} (t Sh_4)^{m_j} \\
&\quad + Aw_4(t^{\times 2} Sh_4)^{m_0} (t Sh_4)^{n_1} (t^{\times 2} Sh_4)^{m_1} \dots (t Sh_4)^{n_j} (t^{\times 2} Sh_4)^{m_j}) \Delta(x)
\end{aligned} \tag{7}$$

where the last sum is taken over the set $\{(n_1, \dots, n_j, m_0, \dots, m_j) : n_1 + \dots + n_j = 3i + 2 - \ell, m_0 + \dots + m_j = \ell, n_k \geq 0, m_k \geq 0\}$.

In order to simplify (7), we will use the facts that

$$Aw_4(t Sh_4)^n = D_n \text{ if } n \geq 0.$$

$$Aw_4(t Sh_4)^n t^{\times 2} Sh_4 = 0 \text{ if } n \geq 1.$$

$$Aw_4(t^{\times 2} Sh_4)^n = \sum_{0 \leq j \leq n} (D_j t^{n-j} \otimes T^j D_{n-j}) D_0 \text{ if } n \geq 0.$$

$$\begin{aligned}
&Aw_4(t^{\times 2} Sh_4)^n (t Sh_4)^m \\
&= \sum_{0 \leq j \leq n} (D_j t^{n-j} \otimes T^j D_{n-j}) D_m + \sum_{\substack{0 \leq j \leq n \\ j+n \text{ is odd}}} (D_{j+1} \otimes T^j D_{n-j}) T D_{m-1} \text{ if } n, m > 0.
\end{aligned}$$

$$Aw_4(t^{\times 2} Sh_4)^{n_0} (t Sh_4)^m (t^{\times 2} Sh_4)^{n_1} = 0 \text{ if } n_0, n_1 > 0, m > 1.$$

$$Aw_4(t^{\times 2} Sh_4)^{n_0} t Sh_4 (t^{\times 2} Sh_4)^{n_1}$$

$$= \sum_{\substack{0 \leq j \leq n_0 \\ j+n_0 \text{ is odd}}} (D_{j+1+n_1} \otimes T^j D_{n_0-j} t^{n_1}) T D_0 \text{ if } n_0, n_1 > 0$$

$$Aw_4(t^{\times 2} Sh_4)^{n_0} (t Sh_4)^{m_1} (t^{\times 2} Sh_4)^{n_1} \dots (t Sh_4)^{m_k} (t^{\times 2} Sh_4)^{n_k} (t Sh_4)^m = 0$$

if $m_i > 1$ for some $1 \leq i \leq k$ or n_j is even for some $1 \leq i < k$.

$$Aw_4(t^{\times 2} Sh_4)^{n_0} t Sh_4 (t^{\times 2} Sh_4)^{n_1} \dots t Sh_4 (t^{\times 2} Sh_4)^{n_{2k}} (t Sh_4)^m$$

$$= \begin{cases} \sum_{\substack{0 \leq j \leq n_0 \\ j+n_0 \text{ is odd}}} (D_{j+k+\sum n_{2j-1}} t \otimes T^j D_{-j+k+\sum n_{2j}}) D_m & \text{if } n_{2k} \text{ is even} \\ \sum_{\substack{0 \leq j \leq n_0 \\ j+n_0 \text{ is odd}}} (D_{j+k+\sum n_{2j-1}} t \otimes T^j D_{-j+k+\sum n_{2j}}) D_m \\ + \sum_{\substack{0 \leq j \leq n_0 \\ j+n_0 \text{ is odd}}} (D_{j+k+1+\sum n_{2j-1}} \otimes T^j D_{-j+k+\sum n_{2j}}) T D_{m-1} & \text{otherwise} \end{cases}$$

for $k, n_0, n_1, \dots, n_{2k} > 0, m \geq 0$ and n_i being odd for $1 \leq i \leq 2k-1$.

$$Aw_4(t^{\times 2} Sh_4)^{n_0} t Sh_4 (t^{\times 2} Sh_4)^{n_1} \dots t Sh_4 (t^{\times 2} Sh_4)^{n_{2k-1}} (t Sh_4)^m$$

$$= \begin{cases} \sum_{\substack{0 \leq j \leq n_0 \\ j+n_0 \text{ is odd}}} (D_{j+k+\sum n_{2j-1}} \otimes T^j D_{-j+k-1+\sum n_{2j}} t) T D_m & \text{if } n_{2k-1} \text{ is even} \\ \sum_{\substack{0 \leq j \leq n_0 \\ j+n_0 \text{ is odd}}} (D_{j+k+\sum n_{2j-1}} \otimes T^j D_{-j+k-1+\sum n_{2j}} t) T D_m \\ + \sum_{\substack{0 \leq j \leq n_0 \\ j+n_0 \text{ is odd}}} (D_{j+k+\sum n_{2j-1}} \otimes T^j D_{-j+k+\sum n_{2j}}) D_{m-1} & \text{otherwise} \end{cases}$$

for $k, n_0, n_1, \dots, n_{2k-1} > 0$, $m \geq 0$ and n_i being odd for $1 \leq i \leq 2k-2$.
where $D_i = Aw(tSh)^n$.

Let us observe that all the summands we will obtain when (7) is simplified, has the form: $\mu c^{\otimes 4}(D_a \otimes D_b) D_{2i+2-\ell} \Delta(x)$ and $\mu c^{\otimes 4}(D_d \otimes D_e) D_{i-k} \Delta(x)$ where c is a $(i+2)$ -cocycle, $a+b=i+\ell$, $d+e=2i+2+k$ and $0 \leq \ell, k \leq i$. Let us recall that if c is an m -cocycle and c' is an n -cocycle, then $c \smile_r c' = \mu(c \otimes c') D_r$ is null if $m < r$ or $n < r$. Taking in mind this last fact, we have that $\ell+2 \geq a$ and $\ell+2 \geq b$. Since $a+b=i+\ell$, then $i-4 \leq \ell \leq i$. On the other hand, $d \leq i+2$ and $e \leq i+2$. Since $d+e=2i+2+k$ then $0 \leq k \leq 2$. Therefore, the possible non-null summands of (7) have the form:

$$\begin{aligned} & \mu c^{\otimes 4}(D_{i+2} \otimes D_{i+2}) D_{i-2} \Delta(x), \mu c^{\otimes 4}(D_{i+1} \otimes D_{i+2}) D_{i-1} \Delta(x), \\ & \mu c^{\otimes 4}(D_{i+2} \otimes D_{i+1}) D_{i-1} \Delta(x), \mu c^{\otimes 4}(D_i \otimes D_{i+2}) D_i \Delta(x), \\ & \mu c^{\otimes 4}(D_{i+1} \otimes D_{i+1}) D_i \Delta(x), \mu c^{\otimes 4}(D_{i+2} \otimes D_i) D_i \Delta(x), \\ & \mu c^{\otimes 4}(D_{i+2} \otimes D_{i-1}) D_{i+1} \Delta(x), \mu c^{\otimes 4}(D_{i+1} \otimes D_i) D_{i+1} \Delta(x), \\ & \mu c^{\otimes 4}(D_i \otimes D_{i+1}) D_{i+1} \Delta(x), \mu c^{\otimes 4}(D_{i-1} \otimes D_{i+2}) D_{i+1} \Delta(x), \\ & \mu c^{\otimes 4}(D_{i-2} \otimes D_{i+2}) D_{i+2} \Delta(x), \mu c^{\otimes 4}(D_{i-1} \otimes D_{i+1}) D_{i+2} \Delta(x), \\ & \mu c^{\otimes 4}(D_i \otimes D_i) D_{i+2} \Delta(x), \mu c^{\otimes 4}(D_{i+1} \otimes D_{i-1}) D_{i+2} \Delta(x), \\ & \mu c^{\otimes 4}(D_{i+2} \otimes D_{i-2}) D_{i+2} \Delta(x), \mu c^{\otimes 4}(D_{i-2} \otimes D_{i+1}) D_{i+3} \Delta(x), \\ & \mu c^{\otimes 4}(D_{i-1} \otimes D_i) D_{i+3} \Delta(x), \mu c^{\otimes 4}(D_i \otimes D_{i-1}) D_{i+3} \Delta(x), \\ & \mu c^{\otimes 4}(D_{i+1} \otimes D_{i-2}) D_{i+3} \Delta(x), \mu c^{\otimes 4}(D_{i-2} \otimes D_i) D_{i+4} \Delta(x), \\ & \mu c^{\otimes 4}(D_{i-1} \otimes D_{i-1}) D_{i+4} \Delta(x), \mu c^{\otimes 4}(D_i \otimes D_{i-2}) D_{i+4} \Delta(x), \\ & \mu c^{\otimes 4}(D_{i-2} \otimes D_{i-1}) D_{i+5} \Delta(x), \mu c^{\otimes 4}(D_{i-1} \otimes D_{i-2}) D_{i+5} \Delta(x), \\ & \mu c^{\otimes 4}(D_{i-2} \otimes D_{i-2}) D_{i+6} \Delta(x). \end{aligned}$$

Using the properties mentioned above, the possible non-null summands of (7) for $\ell = i-2$ are:

$$\begin{aligned} & \mu c^{\otimes 4} Aw_4(t^{\times 2} Sh_4)^{2i+4} (t Sh_4)^{i-2} \Delta(x) = \mu c^{\otimes 4}(D_{i+2} \otimes D_{i+2}) D_{i-2} \Delta(x), \\ & \mu c^{\otimes 4} Aw_4 t^{\times 2} Sh_4 (t Sh_4 t^{\times 2} Sh_4)^{i-3} (t Sh_4)^{i+7} \Delta(x) = \mu c^{\otimes 4}(D_{i-2} \otimes D_{i-2}) D_{i+2} \Delta(x). \end{aligned}$$

The possible non-null summands of (7) for $\ell = i-1$ are:

$$\begin{aligned} & \mu c^{\otimes 4} Aw_4(t^{\times 2} Sh_4)^{2i+3} (t Sh_4)^{i-1} \Delta(x) \\ & = \mu c^{\otimes 4}(D_{i+1} \otimes D_{i+2} + D_{i+2} \otimes D_{i+1}) D_{i-1} \Delta(x), \\ & \mu c^{\otimes 4} Aw_4(t^{\times 2} Sh_4)^{2i+3-j} t Sh_4 (t^{\times 2} Sh_4)^j (t Sh_4)^{i-2} \Delta(x) = 0 \text{ for } 1 \leq j \leq 2i+2, \end{aligned}$$

$$\begin{aligned}
& \mu c^{\otimes 4} Aw_4 t^{\times 2} Sh_4 (t Sh_4 t^{\times 2} Sh_4)^{i-2} (t Sh_4)^{i+5} \Delta(x) \\
&= \mu c^{\otimes 4} ((D_{i-2} \otimes D_{i-1}) D_{i+5} + (D_{i-1} \otimes D_{i-1}) D_{i+4}) \Delta(x), \\
& \mu c^{\otimes 4} Aw_4 (t^{\times 2} Sh_4)^2 (t Sh_4 t^{\times 2} Sh_4)^{i-3} (t Sh_4)^{i+6} \Delta(x) \\
&= \mu c^{\otimes 4} (D_{i-1} \otimes D_{i-2}) D_{i+5} \Delta(x), \\
& \mu c^{\otimes 4} Aw_4 t^{\times 2} Sh_4 (t Sh_4 t^{\times 2} Sh_4)^{i-4} t Sh_4 (t^{\times 2} Sh_4)^2 (t Sh_4)^{i+6} \Delta(x) = 0, \\
& \mu c^{\otimes 4} Aw_4 (t^{\times 2} Sh_4)^3 (t Sh_4 t^{\times 2} Sh_4)^{i-4} (t Sh_4)^{i+7} \Delta(x) = 0, \\
& \mu c^{\otimes 4} Aw_4 t^{\times 2} Sh_4 (t Sh_4 t^{\times 2} Sh_4)^{i-5} t Sh_4 (t^{\times 2} Sh_4)^3 (t Sh_4)^{i+7} \Delta(x) = 0, \\
& \mu c^{\otimes 4} Aw_4 t^{\times 2} Sh_4 t Sh_4 (t^{\times 2} Sh_4)^3 (t Sh_4 t^{\times 2} Sh_4)^{i-5} (t Sh_4)^{i+7} \Delta(x) = 0, \\
& \mu c^{\otimes 4} Aw_4 t^{\times 2} Sh_4 (t Sh_4 t^{\times 2} Sh_4)^{i-5-j} t Sh_4 (t^{\times 2} Sh_4)^3 (t Sh_4 t^{\times 2} Sh_4)^j (t Sh_4)^{i+7} \Delta(x) \\
&= 0 \text{ for } 1 \leq j \leq i-6.
\end{aligned}$$

Finally, the possible non-null summands of (7) for $\ell = i$ are:

$$\begin{aligned}
& \mu c^{\otimes 4} Aw_4 (t^{\times 2} Sh_4)^{2i+2} (t Sh_4)^i \Delta(x) \\
&= \begin{cases} \mu c^{\otimes 4} ((D_i \otimes D_{i+2} + D_{i+1} \otimes D_{i+1} + D_{i+2} \otimes D_i) D_i + (D_{i+1} \otimes D_{i+2}) D_{i-1}) \Delta(x) & \text{if } i \text{ is odd} \\ \mu c^{\otimes 4} ((D_i \otimes D_{i+2} + D_{i+1} \otimes D_{i+1} + D_{i+2} \otimes D_i) D_i + (D_{i+2} \otimes D_{i+1}) D_{i-1}) \Delta(x) & \text{if } i \text{ is even,} \end{cases} \\
& \mu c^{\otimes 4} Aw_4 (t^{\times 2} Sh_4)^{2i+2-j} t Sh_4 (t^{\times 2} Sh_4)^j (t Sh_4)^{i-1} \Delta(x) \\
&= \begin{cases} \mu c^{\otimes 4} (D_{i+1} \otimes D_{i+2}) D_{i-1} \Delta(x) & \text{if } i \text{ is odd} \\ \mu c^{\otimes 4} (D_{i+2} \otimes D_{i+1}) D_{i-1} \Delta(x) & \text{if } j, i \text{ are even} \\ \mu c^{\otimes 4} ((D_{i+2} \otimes D_{i+1}) D_{i-1} + D_{i+2} \otimes D_{i+2}) D_{i-2} \Delta(x) & \text{if } j \text{ is odd, } i \text{ is even} \end{cases} \\
& \text{for } 1 \leq j \leq 2i+1, \\
& \mu c^{\otimes 4} Aw_4 (t^{\times 2} Sh_4)^{2i+2-j-k} t Sh_4 (t^{\times 2} Sh_4)^j t Sh_4 (t^{\times 2} Sh_4)^k (t Sh_4)^{i-2} \Delta(x) \\
&= \begin{cases} \mu c^{\otimes 4} (D_{i+2} \otimes D_{i+2}) D_{i-2} \Delta(x) & \text{if } k, i \text{ are odd} \\ \mu c^{\otimes 4} (D_{i+2} \otimes D_{i+2}) D_{i-2} \Delta(x) & \text{if } k, i \text{ are even} \\ 0 & \text{otherwise} \end{cases} \\
& \text{for } 1 \leq j, k, j+k \leq 2i+1 \text{ and } j \text{ being odd,} \\
& \mu c^{\otimes 4} Aw_4 t^{\times 2} Sh_4 (t Sh_4 t^{\times 2} Sh_4)^{i-1} (t Sh_4)^{i+3} \Delta(x) \\
&= \begin{cases} \mu c^{\otimes 4} ((D_{i-1} \otimes D_i) D_{i+3} + (D_i \otimes D_i) D_{i+2}) \Delta(x) & \text{if } i \text{ is odd} \\ \mu c^{\otimes 4} ((D_i \otimes D_{i-1}) D_{i+3} + (D_i \otimes D_i) D_{i+2}) \Delta(x) & \text{if } i \text{ is even,} \end{cases} \\
& \mu c^{\otimes 4} Aw_4 (t^{\times 2} Sh_4)^2 (t Sh_4 t^{\times 2} Sh_4)^{i-2} (t Sh_4)^{i+4} \Delta(x) \\
&= \begin{cases} \mu c^{\otimes 4} ((D_i \otimes D_{i-2}) D_{i+4} + (D_i \otimes D_{i-1}) D_{i+3}) \Delta(x) & \text{if } i \text{ is odd} \\ \mu c^{\otimes 4} ((D_{i-1} \otimes D_{i-1}) D_{i+4} + (D_i \otimes D_{i-1}) D_{i+3}) \Delta(x) & \text{if } i \text{ is even} \end{cases} \\
& \mu c^{\otimes 4} Aw_4 t^{\times 2} Sh_4 (t Sh_4 t^{\times 2} Sh_4)^{i-3} t Sh_4 (t^{\times 2} Sh_4)^2 (t Sh_4)^{i+4} \Delta(x) \\
&= \begin{cases} \mu c^{\otimes 4} (D_i \otimes D_{i-2}) D_{i+4} \Delta(x) & \text{if } i \text{ is odd} \\ \mu c^{\otimes 4} (D_{i-2} \otimes D_i) D_{i+4} \Delta(x) & \text{if } i \text{ is even,} \end{cases} \\
& \mu c^{\otimes 4} Aw_4 (t^{\times 2} Sh_4)^3 (t Sh_4 t^{\times 2} Sh_4)^{i-3} (t Sh_4)^{i+5} \Delta(x) \\
&= \begin{cases} \mu c^{\otimes 4} ((D_{i-1} \otimes D_{i-2}) D_{i+5} + (D_{i-2} \otimes D_i + D_i \otimes D_{i-2}) D_{i+4}) \Delta(x) & \text{if } i \text{ is odd} \\ \mu c^{\otimes 4} ((D_{i-2} \otimes D_{i-1}) D_{i+5} + (D_{i-2} \otimes D_i + D_i \otimes D_{i-2}) D_{i+4}) \Delta(x) & \text{if } i \text{ is even,} \end{cases} \\
& \mu c^{\otimes 4} Aw_4 (t^{\times 2} Sh_4)^2 (t Sh_4 t^{\times 2} Sh_4)^{i-4} t Sh_4 (t^{\times 2} Sh_4)^2 (t Sh_4)^{i+5} \Delta(x)
\end{aligned}$$

$$\begin{aligned}
&= \begin{cases} \mu c^{\otimes 4}(D_{i-2} \otimes D_{i-1})D_{i+5}\Delta(x) & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even,} \end{cases} \\
&\mu c^{\otimes 4}Aw_4t^{\times 2}Sh_4(tSh t^{\times 2}Sh_4)^{i-4}tSh(t^{\times 2}Sh_4)^3(tSh_4)^{i+5}\Delta(x) \\
&= \begin{cases} \mu c^{\otimes 4}(D_{i-2} \otimes D_i)D_{i+4}\Delta(x) & \text{if } i \text{ is odd} \\ \mu c^{\otimes 4}(D_i \otimes D_{i-2})D_{i+4}\Delta(x) & \text{if } i \text{ is even,} \end{cases} \\
&\mu c^{\otimes 4}Aw_4t^{\times 2}Sh_4tSh(t^{\times 2}Sh_4)^3(tSh t^{\times 2}Sh_4)^{i-4}(tSh_4)^{i+5}\Delta(x) \\
&= \begin{cases} \mu c^{\otimes 4}((D_{i-1} \otimes D_{i-2})D_{i+5} + (D_i \otimes D_{i-2})D_{i+4})\Delta(x) & \text{if } i \text{ is odd} \\ \mu c^{\otimes 4}(D_i \otimes D_{i-2})D_{i+4}\Delta(x) & \text{if } i \text{ is even,} \end{cases} \\
&\mu c^{\otimes 4}Aw_4t^{\times 2}Sh_4(tSh t^{\times 2}Sh_4)^{i-4-j}tSh(t^{\times 2}Sh_4)^3(tSh t^{\times 2}Sh_4)^j(tSh_4)^{i+5}\Delta(x) \\
&= \begin{cases} \mu c^{\otimes 4}((D_{i-1} \otimes D_{i-2})D_{i+5} + (D_i \otimes D_{i-2})D_{i+4})\Delta(x) & \text{if } j, i \text{ are odd} \\ \mu c^{\otimes 4}(D_{i-2} \otimes D_i)D_{i+4}\Delta(x) & \text{if } j \text{ is even, } i \text{ is odd} \\ \mu c^{\otimes 4}((D_{i-2} \otimes D_{i-1})D_{i+5} + (D_{i-2} \otimes D_i)D_{i+4})\Delta(x) & \text{if } j \text{ is odd, } i \text{ is even} \\ \mu c^{\otimes 4}(D_i \otimes D_{i-2})D_{i+4}\Delta(x) & \text{if } j, i \text{ is even} \end{cases} \\
&\text{for } 1 \leq j \leq i-5, \\
&\mu c^{\otimes 4}Aw_4(t^{\times 2}Sh_4)^4(tSh t^{\times 2}Sh_4)^{i-4}(tSh_4)^{i+6}\Delta(x) \\
&= \begin{cases} \mu c^{\otimes 4}((D_{i-2} \otimes D_{i-2})D_{i+6} + (D_{i-2} \otimes D_{i-1})D_{i+5})\Delta(x) & \text{if } i \text{ is odd} \\ \mu c^{\otimes 4}(D_{i-2} \otimes D_{i-1})D_{i+5}\Delta(x) & \text{if } i \text{ is even,} \end{cases} \\
&\mu c^{\otimes 4}Aw_4(t^{\times 2}Sh_4)^3(tSh t^{\times 2}Sh_4)^{i-5}tSh(t^{\times 2}Sh_4)^2(tSh_4)^{i+6}\Delta(x) \\
&= \mu c^{\otimes 4}(D_{i-2} \otimes D_{i-2})D_{i+6}\Delta(x), \\
&\mu c^{\otimes 4}Aw_4(t^{\times 2}Sh_4)^2(tSh t^{\times 2}Sh_4)^{i-5}tSh(t^{\times 2}Sh_4)^3(tSh_4)^{i+6}\Delta(x) \\
&= \begin{cases} 0 & \text{if } i \text{ is odd} \\ \mu c^{\otimes 4}(D_{i-2} \otimes D_{i-1})D_{i+5}\Delta(x) & \text{if } i \text{ is even,} \end{cases} \\
&\mu c^{\otimes 4}Aw_4(t^{\times 2}Sh_4)^2tSh(t^{\times 2}Sh_4)^3(tSh t^{\times 2}Sh_4)^{i-5}(tSh_4)^{i+6}\Delta(x) = 0, \\
&\mu c^{\otimes 4}Aw_4(t^{\times 2}Sh_4)^2(tSh t^{\times 2}Sh_4)^{i-5-j}tSh(t^{\times 2}Sh_4)^3(tSh t^{\times 2}Sh_4)^j(tSh_4)^{i+6}\Delta(x) \\
&= \begin{cases} \mu c^{\otimes 4}((D_{i-2} \otimes D_{i-2})D_{i+6} + (D_{i-2} \otimes D_{i-1})D_{i+5})\Delta(x) & \text{if } j, i \text{ are odd} \\ \mu c^{\otimes 4}(D_{i-2} \otimes D_{i-1})D_{i+5}\Delta(x) & \text{if } j, i \text{ are even} \\ 0 & \text{otherwise} \end{cases} \\
&\text{for } 1 \leq j \leq i-6, \\
&\mu c^{\otimes 4}Aw_4t^{\times 2}Sh_4(tSh t^{\times 2}Sh_4)^{i-5}tSh(t^{\times 2}Sh_4)^4(tSh_4)^{i+6}\Delta(x) = 0, \\
&\mu c^{\otimes 4}Aw_4t^{\times 2}Sh_4tSh(t^{\times 2}Sh_4)^3(tSh t^{\times 2}Sh_4)^{i-6}tSh(t^{\times 2}Sh_4)^2(tSh_4)^{i+6}\Delta(x) \\
&= \begin{cases} \mu c^{\otimes 4}(D_{i-2} \otimes D_{i-2})D_{i+6}\Delta(x) & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even,} \end{cases} \\
&\mu c^{\otimes 4}Aw_4t^{\times 2}Sh_4(tSh t^{\times 2}Sh_4)^{i-6-j}tSh(t^{\times 2}Sh_4)^3(tSh t^{\times 2}Sh_4)^j(tSh t^{\times 2}Sh_4)^2(tSh_4)^{i+6}\Delta(x) \\
&= \begin{cases} \mu c^{\otimes 4}(D_{i-2} \otimes D_{i-2})D_{i+6}\Delta(x) & \text{if } j \text{ is even} \\ 0 & \text{otherwise} \end{cases} \\
&\text{for } 1 \leq j \leq i-7, \\
&\mu c^{\otimes 4}Aw_4t^{\times 2}Sh_4(tSh t^{\times 2}Sh_4)^{i-6}tSh(t^{\times 2}Sh_4)^3tSh(t^{\times 2}Sh_4)^2(tSh_4)^{i+6}\Delta(x)
\end{aligned}$$

$$\begin{aligned}
&= \mu c^{\otimes 4} (D_{i-2} \otimes D_{i-2}) D_{i+6} \Delta(x), \\
&\mu c^{\otimes 4} Aw_4(t^{\times 2} Sh_4)^5 (t Sh t^{\times 2} Sh_4)^{i-5} (t Sh_4)^{i+7} \Delta(x) \\
&= \mu c^{\otimes 4} (D_{i-2} \otimes D_{i-2}) D_{i+6} \Delta(x), \\
&\mu c^{\otimes 4} Aw_4 t^{\times 2} Sh_4 (t Sh t^{\times 2} Sh_4)^{i-6} t Sh (t^{\times 2} Sh_4)^5 (t Sh_4)^{i+7} \Delta(x) = 0, \\
&\mu c^{\otimes 4} Aw_4 t^{\times 2} Sh_4 (t Sh t^{\times 2} Sh_4)^{i-6-j} t Sh (t^{\times 2} Sh_4)^5 (t Sh t^{\times 2} Sh_4)^j (t Sh_4)^{i+7} \Delta(x) \\
&= 0 \text{ for } 1 \leq j \leq i-7, \\
&\mu c^{\otimes 4} Aw_4 (t^{\times 2} Sh_4)^3 (t Sh t^{\times 2} Sh_4)^{i-6} t Sh (t^{\times 2} Sh_4)^3 (t Sh_4)^{i+7} \Delta(x) \\
&= \mu c^{\otimes 4} (D_{i-2} \otimes D_{i-2}) D_{i+6} \Delta(x), \\
&\mu c^{\otimes 4} Aw_4 (t^{\times 2} Sh_4)^3 t Sh (t^{\times 2} Sh_4)^3 (t Sh t^{\times 2} Sh_4)^{i-6} (t Sh_4)^{i+7} \Delta(x) \\
&= \mu c^{\otimes 4} (D_{i-2} \otimes D_{i-2}) D_{i+6} \Delta(x), \\
&\mu c^{\otimes 4} Aw_4 (t^{\times 2} Sh_4)^3 (t Sh t^{\times 2} Sh_4)^{i-6-j} t Sh (t^{\times 2} Sh_4)^3 (t Sh t^{\times 2} Sh_4)^j (t Sh_4)^{i+7} \Delta(x) \\
&= \mu c^{\otimes 4} (D_{i-2} \otimes D_{i-2}) D_{i+6} \Delta(x) \text{ for } 1 \leq j \leq i-7, \\
&\mu c^{\otimes 4} Aw_4 t^{\times 2} Sh_4 t Sh (t^{\times 2} Sh_4)^3 (t Sh t^{\times 2} Sh_4)^{i-7} t Sh (t^{\times 2} Sh_4)^3 (t Sh_4)^{i+7} \Delta(x) \\
&= \begin{cases} \mu c^{\otimes 4} (D_{i-2} \otimes D_{i-2}) D_{i+6} \Delta(x) & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even,} \end{cases} \\
&\mu c^{\otimes 4} Aw_4 t^{\times 2} Sh_4 (t Sh t^{\times 2} Sh_4)^{i-7} t Sh (t^{\times 2} Sh_4)^3 t Sh (t^{\times 2} Sh_4)^3 (t Sh_4)^{i+7} \Delta(x) \\
&= \mu c^{\otimes 4} (D_{i-2} \otimes D_{i-2}) D_{i+6} \Delta(x), \\
&\mu c^{\otimes 4} Aw_4 t^{\times 2} Sh_4 (t Sh t^{\times 2} Sh_4)^{i-7-j} t Sh (t^{\times 2} Sh_4)^3 (t Sh t^{\times 2} Sh_4)^j t Sh (t^{\times 2} Sh_4)^3 (t Sh_4)^{i+7} \Delta(x), \\
&= \begin{cases} \mu c^{\otimes 4} (D_{i-2} \otimes D_{i-2}) D_{i+6} \Delta(x) & \text{if } j \text{ is even} \\ 0 & \text{otherwise} \end{cases} \\
&\text{for } 1 \leq j \leq i-8.
\end{aligned}$$

Now, using property (1), the simplified expression for (7) mod 2 is:

$$\begin{aligned}
&\mu c^{\otimes 4} ((D_0 \otimes D_2 + D_2 \otimes D_0 + D_1 \otimes D_1) D_0 + (D_0 \otimes D_0) D_2) \Delta(x) \\
&= \delta(c \smile_0 c \smile_1 c \smile_2 c) + c \smile_1 c \smile_0 c \smile_1 c + c \smile_0 c \smile_2 c \smile_0 c \text{ for } i = 0. \\
&\mu c^{\otimes 4} ((D_1 \otimes D_3 + D_3 \otimes D_1 + D_2 \otimes D_2) D_1 + (D_1 \otimes D_1) D_3) \Delta(x) \\
&= \delta(c \smile_1 c \smile_2 c \smile_3 c) + c \smile_2 c \smile_1 c \smile_2 c + c \smile_1 c \smile_3 c \smile_1 c \text{ for } i = 1. \\
&\mu c^{\otimes 4} ((D_2 \otimes D_4 + D_4 \otimes D_2 + D_3 \otimes D_3) D_2 + (D_2 \otimes D_2) D_4 \\
&\quad + (D_0 \otimes D_2 + D_2 \otimes D_0) D_6) \Delta(x) \\
&= \delta(c \smile_2 c \smile_3 c \smile_4 c + c \smile_0 c \smile_7 c \smile_2 c) + c \smile_3 c \smile_2 c \smile_3 c \\
&\quad + c \smile_2 c \smile_4 c \smile_2 c \text{ for } i = 2. \\
&\mu c^{\otimes 4} ((D_3 \otimes D_5 + D_5 \otimes D_3 + D_4 \otimes D_4) D_3 + (D_3 \otimes D_3) D_5 \\
&\quad + (D_2 \otimes D_3 + D_3 \otimes D_2) D_6 + (D_3 \otimes D_1 + D_1 \otimes D_3) D_7) \Delta(x) \\
&= \delta(c \smile_3 c \smile_4 c \smile_5 c + c \smile_2 c \smile_7 c \smile_3 c + c \smile_3 c \smile_8 c \smile_1 c) \\
&\quad + c \smile_3 c \smile_5 c \smile_3 c + c \smile_4 c \smile_3 c \smile_4 c \text{ for } i = 3. \\
&\mu c^{\otimes 4} ((D_4 \otimes D_6 + D_6 \otimes D_4 + D_5 \otimes D_5) D_4 + (D_4 \otimes D_4) D_6) \Delta(x) \\
&= \delta(c \smile_4 c \smile_5 c \smile_6 c) + c \smile_5 c \smile_4 c \smile_5 c + c \smile_4 c \smile_6 c \smile_4 c \text{ for } i = 4. \\
&\mu c^{\otimes 4} ((D_5 \otimes D_7 + D_7 \otimes D_5 + D_6 \otimes D_6) D_5 + (D_5 \otimes D_5) D_7
\end{aligned}$$

$$\begin{aligned}
& +(D_4 \otimes D_5 + D_5 \otimes D_4)D_8 + (D_4 \otimes D_3 + D_3 \otimes D_4)D_{10})\Delta(x) \\
& = \delta(c \smile_5 c \smile_6 c \smile_7 c + c \smile_4 c \smile_9 c \smile_5 c + c \smile_4 c \smile_{11} c \smile_3 c) \\
& \quad + c \smile_5 c \smile_7 c \smile_5 c + c \smile_6 c \smile_5 c \smile_6 c \text{ for } i = 5. \\
& \mu c^{\otimes 4}((D_i \otimes D_{i+2} + D_{i+2} \otimes D_i + D_{i+1} \otimes D_{i+1})D_i + (D_i \otimes D_i)D_{i+2})\Delta(x) \\
& = \delta(c \smile_i c \smile_{i+1} c \smile_{i+2} c) + c \smile_{i+1} c \smile_i c \smile_{i+1} c + c \smile_i c \smile_{i+2} c \smile_i c \\
& \text{for } i \geq 6, i \text{ being even, } \frac{i-6}{2} \text{ being odd.} \\
& \mu c^{\otimes 4}((D_i \otimes D_{i+2} + D_{i+2} \otimes D_i + D_{i+1} \otimes D_{i+1})D_i + (D_i \otimes D_i)D_{i+2} \\
& \quad + (D_i \otimes D_{i-2} + D_{i-2} \otimes D_i)D_{i+4} \\
& = \delta(c \smile_i c \smile_{i+1} c \smile_{i+2} c + c \smile_i c \smile_{i+5} c \smile_{i-2} c) \\
& \quad + c \smile_{i+1} c \smile_i c \smile_{i+1} c + c \smile_i c \smile_{i+2} c \smile_i c \\
& \text{for } i \geq 6 \text{ and } i, \frac{i-6}{2} \text{ being even.} \\
& \mu c^{\otimes 4}((D_i \otimes D_{i+2} + D_{i+2} \otimes D_i + D_{i+1} \otimes D_{i+1})D_i + (D_i \otimes D_i)D_{i+2} \\
& \quad + (D_{i-1} \otimes D_i + D_i \otimes D_{i-1})D_{i+3} + (D_i \otimes D_{i-2} + D_{i-2} \otimes D_i)D_{i+4} \\
& \quad + (D_{i-1} \otimes D_{i-2} + D_{i-2} \otimes D_{i-1})D_{i+5})\Delta(x) \\
& = \delta(c \smile_i c \smile_{i+1} c \smile_{i+2} c + c \smile_i c \smile_{i+4} c \smile_{i-1} c + c \smile_i c \smile_{i+5} c \smile_{i-2} c \\
& \quad + c \smile_{i-1} c \smile_{i+6} c \smile_{i-2} c) + c \smile_{i+1} c \smile_i c \smile_{i+1} c + c \smile_i c \smile_{i+2} c \smile_i c \\
& \text{for } i \geq 7 \text{ and } i, \frac{i-7}{2} \text{ being odd.} \\
& \mu c^{\otimes 4}((D_i \otimes D_{i+2} + D_{i+2} \otimes D_i + D_{i+1} \otimes D_{i+1})D_i + (D_i \otimes D_i)D_{i+2} \\
& \quad + (D_{i-1} \otimes D_i + D_i \otimes D_{i-1})D_{i+3} + (D_i \otimes D_{i-2} + D_{i-2} \otimes D_i)D_{i+4} \\
& \quad + (D_{i-1} \otimes D_{i-2} + D_{i-2} \otimes D_{i-1})D_{i+5})\Delta(x) \\
& = \delta(c \smile_i c \smile_{i+1} c \smile_{i+2} c + c \smile_i c \smile_{i+4} c \smile_{i-1} c + c \smile_i c \smile_{i+5} c \smile_{i-2} c \\
& \quad + c \smile_{i-1} c \smile_{i+6} c \smile_{i-2} c) + c \smile_{i+1} c \smile_i c \smile_{i+1} c + c \smile_i c \smile_{i+2} c \smile_i c \\
& \text{for } i \geq 7, i \text{ being odd, } \frac{i-7}{2} \text{ being even.}
\end{aligned}$$

This completes the proof.